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ISOMETRIC HOMOTOPY AND CODIMENSION-TWO ISOMETRIC IMMERSIONS OF THE n-SPHERE INTO EUCLIDEAN SPACE

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Introduction

It is a well-known result in differential geometry that any isometric immersion of the constant curvature sphere $S^n(\subset \mathbf{R}^{n+1})$ into flat Euclidean space $\mathbf{R}^{n+1}$ is the restriction of a rigid motion of $\mathbf{R}^{n+1}$. Easy examples in higher codimensions show that the obvious generalization is false. For example, the "cylindrical rolling" $g\colon S^n \to \mathbf{R}^{n+1}$ defined by

$$(p_1, \cdots, p_{n+1}) \to (\cos p_1, \sin p_1, p_2, \cdots, p_{n+1}),$$

where $p_1^2 + \cdots + p_{n+1}^2 = C$, is not the restriction of a rigid motion of $\mathbf{R}^{n+2}$. However, g can be deformed into a restriction by "unrolling" it, i.e., by a homotopy through isometric immersions. In this paper, we prove that every isometric immersion of S^n into $\mathbf{R}^{n+2}$ ($n \geqslant 3$) is *isometrically homotopic* to the restriction of a rigid motion of $\mathbf{R}^{n+2}$ (Theorem 3.1). Clearly these restrictions are the inclusions of S^n into hyperplanes $\mathbf{R}^{n+1} \subset \mathbf{R}^{n+2}$. A similar result is obtained for isometric immersions of $\mathbf{R}^m$ into $\mathbf{R}^n$ with zero normal curvature (Theorem 3.3). Both results are corollaries of a more general theorem (Theorem 2.1). Namely, a 1-parameter family of Riemannian manifolds produces a homotopy through isometric immersions. This is essentially a 1-parameter version of the existence theorem for isometric immersions (cf. [1, Theorem 5, p. 202] or [13]).

The cylindrical rolling above obviously extends to an isometric immersion of $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+2}$. However, there do exist isometric immersions $f\colon S^n \to \mathbf{R}^{n+2}$ which do not isometrically extend, not even to a neighborhood of $S^n \subset \mathbf{R}^{n+1}$, [7], [15]. In contrast [15], if $f_0, f_1\colon S^n \to \mathbf{R}^{n+2}$, $n \geqslant 3$, are isometric immersions, then an isometric homotopy $\{f_t\}$ can be chosen so that f_t extends isometrically to a neighborhood of S^n, for $t \neq 0$, 1 (the neighborhood depends on t and may be chosen to contain the disc $D^{n+1} \subset \mathbf{R}^{n+1}$). Finally, every isometric immersion $f\colon S^n \to \mathbf{R}^{n+2}$, $n \geqslant 1$, extends to an immersion (not

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necessarily isometric) of a neighborhood of D^{n+1} (Corollary 3.2).

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1. Preliminaries

Throughout this paper, all mappings, manifolds, bundles, etc., will be differentiable of class C^∞, unless otherwise indicated. We assume that S^n has constant curvature 1, although none of our results are affected if 1 is replaced by any positive constant C.

Let M be a Riemannian n-manifold and let ϕ: $M \to \mathbf{R}^q$ be an isometric immersion with normal bundle $\nu(\phi)$. This bundle has an induced bundle metric obtained by regarding the fibres as subspaces of $\mathbf{R}^q$. Let $\overline{\nabla}$ (resp. ∇) be the Riemannian connection of $\mathbf{R}^q$ (resp. M). If X and Y (resp. N) are tangent (resp. normal) vector fields on M, then

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y), \quad \overline{\nabla} = dXN = A_X N + D_X N,$$

where the right side is the decomposition into tangent and normal components. The second fundamental tensor B depends only on X and Y at each point, and hence is a linear mapping B: $TM \otimes TM \to \nu(\phi)$. The second fundamental form A depends only on X and N at each point, and hence is a linear mapping A: $TM \otimes \nu(\phi) \to TM$. These are related by $\langle B(X, Y), N\rangle = \langle A_X N, Y\rangle$, where $\langle\ ,\ \rangle$ denotes the metric in both TM and $\nu(\phi)$. It is easy to see that A is symmetric, or equivalently, $B(X, Y) = B(Y, X)$. D is the normal connection and satisfies the compatibility condition

$$X \cdot \langle N_1, N_2\rangle = \langle D_X N_1, N_2\rangle + \langle N_1, D_X N_2\rangle.$$

The curvature form of D is given by the usual formula

$$\tilde{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[X,Y]} N.$$

The following equations are necessary conditions for the existence of ϕ:

(Gauss)
$$R(X, Y)Z = A_X B(Y, Z) - A_Y B(X, Z),$$
$$\tilde{R}(X, Y)N = B(A_X N, Y) - B(X, A_Y N);$$

(Codazzi-Mainardi)
$$\nabla_X A_Y N - \nabla_Y A_X N - A_{[X,Y]} N = A_Y D_X N - A_X D_Y N,$$

where R is the Riemannian curvature tensor on M, and Z is a tangent field.

The existence theorem states that the above conditions are sufficient (see especially [13]).

Existence theorem 1.1. *Let M be a simply connected Riemannian n-manifold with a Riemannian m-plane bundle ν over M equipped with a second fundamental form A, an associated second fundamental tensor B, and a compatible normal connection D (compatible with the Riemannian metric on ν). If the Gauss and Codazzi-Mainardi equations are satisfied, then M can be isometrically immersed in $\mathbf{R}^{n+m}$ with normal bundle ν, normal connection D, and second fundamental form A.*

The rigidity theorem states that an isometric immersion is essentially determined by its Riemannian data.

Rigidity theorem 1.2. *Let ϕ, ϕ': $M \to \mathbf{R}^{n+m}$ be isometric immersions of a connected Riemannian n-manifold (not necessarily simply connected) with normal bundles ν, ν' equipped as above with bundle metrics, connections, and second fundamental forms. Suppose that there is an isometry s: $M \to M$ which can be covered by a bundle map $\bar{s}$: $\nu \to \nu'$ preserving the bundle metrics, the connections, and the second fundamental forms. Then there is a rigid motion S of $\mathbf{R}^{n+m}$ such that $S \circ \phi = \phi' \circ s$.*

2. Isometric homotopy

Let M and M' be Riemannian manifolds. A mapping H: $M \times I \to M'$ ($I = [0, 1]$) is an *isometric homotopy* if $H(-, t)$: $M \to M'$ is an isometric immersion for all t. In this paper, we will consider only the case where M' is a Euclidean space. Recall that C^∞ differentiability is assumed everywhere; as usual, a mapping defined on a manifold with boundary is C^∞ if it C^∞ extends to an open neighborhood.

Let ν be a Riemannian m-vector bundle over M with projection π: $\nu \to M$. We use $\langle \, , \rangle$ to denote the metric on both TM and ν since no confusion seems likely.

Definition. The 3-tuple (A, B, D) is a *1-parameter family of Riemannian data* on ν, or on M, if the following conditions are satisfied:

(1) A is a 1-parameter family of second fundamental forms, i.e., there is a mapping A: $(\nu \otimes TM) \times I \to TM$, and for each t, A^t is a second fundamental form. Define the second fundamental tensor B: $(TM \otimes TM) \times I \to \nu$ by the usual formula, for each t.

(2) D is a 1-parameter family of bundle connections on ν, each compatible with $\langle \, , \rangle$, i.e., by choosing a local orthonormal framing of ν, there is an associated 1-parameter family of connection 1-forms ω_{ij}: $TM \times I \to \mathbf{R}$ satisfying $\omega_{ij} = -\omega_{ji}$, for each $1 \leqslant i, j \leqslant m$.

(3) For each t, the data (A^t, B^t, D^t) satisfy the Gauss and Codazzi-Mainardi equations.

We will frequently use the notation $\{A^t\}$, $\{B^t\}$, and $\{D^t\}$ for A, B, and D.

Let M be a simply connected Riemannian n-manifold and let ν be a Riemannian m-plane bundle. If (A, B, D) is a 1-parameter family of Riemannian data on ν, then for each $t \in I$, we can apply the existence theorem to obtain an isometric immersion f_t: $M \to \mathbf{R}^{n+m}$ realizing ν, A^t, B^t, and D^t. Furthermore, if we choose $p \in M$ and fix orthonormal frames $\{X_1, \cdots, X_n\} \subset T_pM$ and $\{N_1, \cdots, N_m\} \subset \nu_p$, then by the rigidity theorem, f_t is uniquely determined by the conditions

$$\begin{aligned} &f_t(p) = \text{origin of } \mathbf{R}^{n+m}, \\ &df_t X_i = e_i, \qquad i = 1, \cdots, n, \\ &N_i = e_{n+i}, \qquad i = 1, \cdots, m, \end{aligned} \tag{2.1}$$

where $\{e_1, \cdots, e_{n+m}\}$ is the standard basis of $\mathbf{R}^{n+m}$. We will prove

Theorem 2.1. *Let M be a connected, simply connected Riemannian n-manifold, and let ν be a Riemannian m-plane bundle over M. Let (A, B, D) be a 1-parameter family of Riemannian data on ν, and let f_t: $M \to \mathbf{R}^{n+m}$ be the unique isometric immersion realizing ν, A^t, B^t, and D^t, and satisfying* (2.1). *Then the map $\{f_t\}$: $M \times I \to \mathbf{R}^{n+m}$, defined by $\{f_t\}(x, s) = f_s(x)$, is an isometric homotopy.*

Proof. It is sufficient to show that $\{f_t\}$ is C^∞ differentiable. Since the existence theorem plays an essential role, we begin with an outline of its proof for each f_t. More details can be found in [13], [14]. Let $T\nu \approx H^t \oplus V^t$ be the decomposition into horizontal and vertical subbundles defined by the connection D^t. We define a (possibly singular) metric g_t on $T\nu$ to be the orthogonal direct sum of the metric on V^t induced by the identification $V^t \approx \pi^*\nu$ from the given metric on ν, and the (possibly singular) metric $\langle\, , \rangle_t$ on H^t given by

$$\langle \tilde{X}, \tilde{Y} \rangle_t = \langle X + A_X^t N, Y + A_Y^t N \rangle,$$

where $\tilde{X}$, $\tilde{Y} \in T_N\nu$ are horizontal, $\tilde{X} = \pi_* X$, $\tilde{Y} = \pi_* Y$, and $\langle\, , \rangle$ is the metric on TM. Actually, there is a tubular neighborhood W^t of the zero section in ν on which g_t is positive definite.

For a fixed t, the following two lemmas complete the proof of the existence theorem.

Lemma 2.2. *If the Gauss and Codazzi-Mainardi equations are satisfied by A^t, B^t, and D^t, then the metric g_t defined on W^t is flat.*

Lemma 2.3. *Suppose M_1 and M_2 are flat Riemannian n-manifolds with M_1 simply connected and M_2 complete. Then there is an isometric immersion of M_1 into M_2.*

To determine the effect of varying t, first note that the differentiability of A and D implies, by construction, that the 1-parameter family of (possibly singular) metrics $\{g_t\}$: $(T\nu \otimes T\nu) \times I \to \mathbf{R}$ is C^∞. Since I is compact, it follows that there is a tubular neighborhood W of the zero section in ν on which g_t is positive definite for all $t \in I$. Hence W possesses a 1-parameter family of flat metrics.

Let F_t: $W \to \mathbf{R}^{n+m}$ be an isometric immersion given by Lemma 2.3 which realizes the metric g_t. According to the rigidity theorem, F_t is uniquely determined by the following data at $p \in M$, analogous to the data (2.1).

$$\begin{aligned} &F_t(p) = \text{origin} \in \mathbf{R}^{n+m}, \\ &dF_t(X_i) = e_i, \qquad i = 1, \cdots, n, \\ &dF_t(N_i) = e_{n+i}, \qquad i = 1, \cdots, m. \end{aligned} \tag{2.2}$$

In this case $F_{t|_M} = f_t$. Hence the theorem follows once we establish the C^∞ differentiability of $\{F_t\}$.

Each F_t is essentially a solution to the geodesic system of differential equations. To see this, let $U_t = \exp^{-1}(F_t W)$, where exp: $T_0\mathbf{R}^{n+m} \to \mathbf{R}^{n+m}$ is the usual exponential map. By identifying $T_p W$ with $T_0\mathbf{R}^{n+m}$ according to the data (2.2), we may consider a modified exponential map $\exp_t$: $U_t \to W$ (modified in that we allow broken geodesics). Since W is simply connected and flat, it follows that $\exp_t$ is a well-defined isometric diffeomorphism. Let $U = \{(x, t) | x \in U_t,\ t \in I\}$ and define E: $U \to W \times I$ by $E(x, t) = (\exp_t(x), t)$. The C^∞ differentiability of E follows from the theory of differential equations, namely, the C^∞ dependence of solutions (geodesics), on all parameters (see, for example, [2, Theorem 1, p. 335]). In particular, E is a diffeomorphism. Now consider the composition

$$W \times I \xrightarrow{E^{-1}} U \xrightarrow{i} T_0\mathbf{R}^{n+m} \xrightarrow{\exp} \mathbf{R}^{n+m},$$

where i is the inclusion $U_t \hookrightarrow T_0\mathbf{R}^{n+m}$ for each t. This composition is C^∞, and must be $\{F_t\}$ according to the rigidity theorem.

Remark. Szczarba [12, p. 39] indicates how his proof of the existence and rigidity theorems can be adapted to the study of isometric immersions into spheres and hyperbolic space. In a similar way, the proof of Theorem 2.1 can be adapted to the study of isometric homotopies in spheres and hyperbolic spaces. Of course, we would have to suitably modify the Gauss and Codazzi-Mainardi equations. The details have not been carried out.

3. Spheres and cylinders

We will apply the results of the previous section to obtain the following rigidity-type results for isometric immersions.

Theorem 3.1. *Let f: $S^n \to \mathbf{R}^{n+2}$ ($n \geqslant 3$) be an isometric immersion of the* *constant curvature sphere into Euclidean space. Let* $i\colon S^n \hookrightarrow \mathbf{R}^{n+1} \hookrightarrow \mathbf{R}^{n+2}$ *be the standard inclusion of* S^n *into a hyperplane of* $\mathbf{R}^{n+2}$. *Then there is an isometric homotopy* $\{f_t\}$ *with* $f_1 = f$ *and* $f_0 = i$.

Corollary 3.2. *The Smale invariant* $\Omega(f, i)$ *is zero, for all n. In particular, f extends to an immersion of the* $(n+1)$*-disc* D^{n+1}.

Proof of Corollary. For $n \geqslant 3$, from [12, Theorem A] and the remark immediately following it, it follows that $\Omega(f, i) = 0$. Now [12, Theorem E] states that f extends to an immersion of D^{n+1} if and only if $\Omega(f, i) = 0$.

For $n = 2$, it is shown in [9, Theorem 8.2] that the regular homotopy class of an immersion $h\colon S^2 \to \mathbf{R}^4$ depends only on the Euler class χ of the normal bundle $\nu(h)$. If, in addition, h is an isometry, then it is not difficult to show that the mean curvature field of h is a nowhere zero section of ν. Hence $\chi(\nu) = 0$ and h is regularly homotopic to an inclusion $i\colon S^2 \hookrightarrow \mathbf{R}^3 \hookrightarrow \mathbf{R}^4$. Thus [12, Theorem E] implies that h extends to an immersion of D^3.

Finally, the case $n = 1$ follows from [12, Theorem B].

Theorem 3.3. *Let* $f\colon \mathbf{R}^m \to \mathbf{R}^n$ *be an isometric immersion such that the normal curvature is zero, and let* $i\colon \mathbf{R}^m \hookrightarrow \mathbf{R}^n$ *be the standard inclusion. Then there is an isometric homotopy* $\{f_t\}$ *with* $f_1 = f$ *and* $f_0 = i$.

Corollary 3.4. *If* $f\colon \mathbf{R}^m \to \mathbf{R}^{m+1}$ *is an isometric immersion, then f is isometrically homotopic to* $i\colon \mathbf{R}^m \hookrightarrow \mathbf{R}^{m+1}$.

Proof of Corollary. Since the normal bundle $\nu(f)$ is one-dimensional, it is immediate that the normal curvature is zero. q.e.d.

A result of Hartman and Nirenberg [5] states that any isometric immerson of $\mathbf{R}^m$ into $\mathbf{R}^{m+1}$ is a cylinder erected over a plane curve. In Corollary 3.4, the homotopy unrolls the cylinder into a hyperplane. Similarly, a result of O'Neill [11] states that if $f\colon \mathbf{R}^m \to \mathbf{R}^n$ has constant relative nullity (in the sense of Chern and Kuiper), and the normal curvature is zero, then f is a generalized cylinder, i.e., as a Riemannian product, $f = \bar{f} \times 1\colon \mathbf{R}^{m-k} \times \mathbf{R}^k \to \mathbf{R}^{n-k} \times \mathbf{R}^k$. In Theorem 3.3, the homotopy unrolls f, or more particularly, $\bar{f}$.

Proof of Theorem 3.3. Let A^1, B^1, D^1 be the Riemannian data on the normal bundle $\nu(f) = \nu$. We construct a 1-parameter family of Riemannian data on ν by setting $A^t = tA^1$, $B^t = tB^1$, and $D^t = D^1$ (not tD^1). Using the hypothesis $R(X, Y)Z = 0 = \tilde{R}(X, Y)N$, these data are easily seen to satisfy the Gauss and Codazzi-Mainardi equations. Hence, by Theorem 2.1, f is isometrically homotopic to an isometric immersion $f_0\colon \mathbf{R}^m \to \mathbf{R}^n$ with Riemannian data $A^0 = B^0 = 0$ and $D^0 = D^1$. Since $B^0 = 0$, the image of f_0 is totally geodesic and thus is an m-plane in $\mathbf{R}^n$. If this m-plane needs to be relocated in $\mathbf{R}^n$, then we use a 1-parameter family of rigid motions of $\mathbf{R}^n$.

Proof of Theorem 3.1. The construction of the 1-parameter family of Riemannian data is slightly more complicated here than in the flat case. For

$n \geqslant 3$, there exist two global orthonormal sections N_1, N_2 of the normal bundle $\nu(f)$ which satisfy

$$AN_1 = \text{Identity}, \quad \text{rank}\, AN_2 \leqslant 1.$$

(Henke [6] and Erbacher [4] independently obtained continuous normals for $n \geqslant 4$, and Moore [10] extended to $n \geqslant 3$ and proved C^∞ differentiability.)

For each t, we define Riemannian data on ν (viewed as an abstract 2-plane bundle with a framing N_1, N_2) as follows

$$A^tN_1 = A^1N_1, \quad A^tN_2 = tA^1N_2, \quad D^t = tD^1,$$

where the superscript 1 refers to the data of f. B^t is defined by A^t as in §1. Since these data satisfy the Gauss and Codazzi-Mainardi equations for $t = 1$, it easily follows that these equations are satisfied for all t. By Theorem 2.1, there is an isometric homotopy $\{f_t\}$ with $f_1 = f$. At $t = 0$, we see that N_2 is parallel, i.e., $A^0N_2 = D^0N_2 = 0$, and hence f_0 is an inclusion of S^n into a hyperplane. If necessary, this hyperplane can then be relocated in $\mathbf{R}^{n+2}$ by a 1-parameter family of rigid motions of $\mathbf{R}^{n+2}$. q.e.d.

We conclude with some remarks about isometric immersion $f: S^n \to \mathbf{R}^{n+2}$, $n \geqslant 3$. The proofs are essentially given in [15].

Although f need not extend isometrically to a neighborhood of S^n, it is close in several respects. As mentioned in the introduction, there is an isometric homotopy $\{f_t\}$ so that f_t does extend to a neighborhood for $t < 1$. Furthermore, f extends to a unique continuous map $F: D^{n+1} \to \mathbf{R}^{n+2}$ such that F is a C^∞ isometric immersion on the interior of D^{n+1}. Also, if G is the nonumbillic set of f, and U_0 is the interior of the umbillic set, then f extend isometrically to a neighborhood of $G \cup U_0 \subset S^n \subset \mathbf{R}^{n+1}$. So it is on the boundary of the umbillic set where f misbehaves. The proofs of these extension results consist of extending the Riemannian data of f, and then applying the existence and rigidity theorems.

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TEXAS A & M UNIVERSITY